Solutions EMGMT - Exercises 10 October 2006

1.

$$\sum_{i=0}^{n} 2i = 2\sum_{i=0}^{n} i = 2(0 + \sum_{i=1}^{n} i) = 2(\sum_{i=1}^{n} i) = 2(\frac{n}{2}(n+1)) = n(n+1)$$

2. (a) Number of operations in algorithm arrayMax(A,n):

line 1: indexing + assignment: 2
line 2: subtraction + comparison; n times: $2n$
line 3: subtraction + comparison: for a given i: $n - i + 1$ times, so:

$$\sum_{i=1}^{n-1} 2(n-i+1) = 2\sum_{i=1}^{n-1} n-i+1 = 2\sum_{i=2}^{n} i = 2(\sum_{i=1}^{n} i - 1) =$$

$$2(\frac{n}{2}(n+1) - 1) = n(n+1) - 2$$

line 4: 2 x indexing + comparison; for a given i: $n-i$ times, so total:

$$\sum_{i=1}^{n-1} 3(n-i) = 3\sum_{i=1}^{n-1} n-i = 3\sum_{i=1}^{n-1} i = 3(\frac{n}{2}(n-1)) = \frac{3}{2}n(n-1)$$

line 5: indexing + comparison; for a given i: $n - i$ times, so total:

$$\sum_{i=1}^{n-1} 2(n-i) = n(n-1)$$

line 6: indexing + assignment; for a given i: $n-i$ times, so: $n(n-1)$
End of loop line 3: addition + assignment; for a given i: $n - i + 1$ times, so total (see summation line 3):

$$\sum_{i=1}^{n-1} 2(n-i+1) = n(n+1) - 2$$

End of loop line 2: addition + assignment: $2n$
line 7: return: 1
TOTAL:

$$2+2n+n(n+1)-2+\frac{3}{2}n(n-1)+n(n-1)+n(n-1)+n(n+1)-2+2n+1 =$$

$$n\left(4 + 2(n+1) + \frac{3}{2}(n-1)\right) - 1$$

(b) Number of operations in algorithm weird(A,n):

line 1: comparison; $n+1$ times: $n+1$
line 2: indexing + assignment; n times: $2n$
line 3: assignment; n times: n
line 4: comparison; for a given i: at most $\log n + 1$ times (this is an overestimate; the precise amount is $\log(n-i)+1$, but this would make the calculations too difficult for the goal of this exercise), so:

$$\sum_{i=1}^{n} \log n + 1 = n \log n + n$$

line 5: comparison; $n \log n$ times: $n \log n$
line 6: 3 x indexing + 2 x addition + assignment; for a given i: $4n \log n$ times, so: $6 \cdot (4n \log n) = 24n \log n$
line 7: multiplication + assignment; $n \log n$ times: $2n \log n$
End of loop line 4: no overhead
End of loop line 1: addition + assignment: $2(n+1)$
line 8: return: 1
TOTAL:

$$n+1+2n+n+n\log n+n+n\log n+24n\log n+2n\log n+2(n+1)+1$$

$$= 28n \log n + 7n + 4$$

3. (a)

$$n^3 > 4n^2 + 60n$$

$$\Rightarrow n^3 - 4n^2 - 60n > 0$$

$$\Rightarrow n(n-10)(n+6) > 0$$

This is true if all three factors are positive, or if two are negative and one is positive, so $-6 < n < 0$ or $n < 10$. True for all $n \geq n_0$ if $n_0 = 11$.

(b)

$$8n \log n < 2n^2$$

$$\Rightarrow 4 \log n < n$$

Try for small n that are powers of 2;

$$n = 4 : 8 \not< 4$$

$$n = 8 : 12 \not< 8$$

$$n = 16 : 16 \not< 16$$

True for all $n \geq n_0$ if $n_0 = 17$.

(c)

$$2^n > n^4$$
$$\Rightarrow \log 2^n > \log n^4$$
$$\Rightarrow n > 4 \log n$$

See b).

4. (a) $c = 160, n_0 = 1$
   (b) $c = 32, n_0 = 1$
   (c) $n_0 = 16, c = \frac{1}{10}$

5. $f(n) = n^4 \log n$

6. 2^{10} — $O(1)$
   $2^{\log n} = n, 4n, 3n + 100 \log n$ — $O(n)$,
   $n \log n, 4n \log n + 2n$ — $O(n \log n)$
   $n^2 + 10n$ — $O(n^2)$
   n^3 — $O(n^3)$
   2^n — $O(2^n)$

7. (a) Choose $c = 11$ and $n_0 = 1$. For all $n \geq n_0$:

$$2n^3 + 9n^2 < 11n^3 = c \cdot n^3$$

   (b) Choose $c = \frac{1}{9}$ and $n_0 = 2$. For all $n \geq n_0$:

$$\frac{1}{8} n \log n \geq \frac{1}{9} n \log n = c \cdot n \log n$$

   (c) big-Oh: choose $c = 4$ and $n_0 = 1$. For all $n \geq n_0$:

$$2^{n+2} - n = 4 \cdot 2^n - n < 4 \cdot 2^n = c \cdot 2^n$$

   big-Omega: choose $c = 1$ and $n_0 = 1$. For all $n \geq n_0$:

$$2^{n+2} - n = 4 \cdot 2^n - n > 2^n$$

   (This inequality is true if $3 \cdot 2^n > n$, which indeed holds for all $n \geq 1$.)

8. Given: There are a $c > 0$ and an $n_0 \geq 1$ such that for all $n \geq n_0$:

$$d(n) \leq c \cdot f(n)$$

   To prove: there exists a $c' > 0$ and an $n_0' \geq 1$ such that for all $n \geq n_0'$:

$$a \cdot d(n) \leq c' \cdot f(n)$$

   We choose $c' = a \cdot c$. Then for all $n \geq n_0$:

$$a \cdot d(n) \leq a \cdot c \cdot f(n) = c' \cdot f(n).$$

9. False, because we can find a counterexample. Take $d(n) = 5n, e(n) = 2n, f(n) = n + 1$, and $g(n) = n$. Now $d(n) - e(n) = 3n$, but this is not $O(n + 1 - n) = O(1)$.

10. (a) $O(n^2)$
    (b) $O(n \log n)$